High-Energy Nucleon-Nucleon Scattering*

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The nucleon-nucleon problem is discussed from the standpoint of analyticity in angular momentum. A unique continuation of the partial-wave helicity amplitudes is given. The high-energy nucleon-nucleon problem is then considered from the point of view of the various Regge poles that have the same quantum numbers as the nucleon-antinucleon channel. In particular, the contribution of these Regge trajectories to nucleon-antinucleon scattering and, hence, their contribution by crossing to nucleon-nucleon scattering is given. The resulting formulas should be adequate to describe the total cross section and angular distribution for energies greater than approximately 3 BeV in the laboratory system.

1. INTRODUCTION

THE nonrelativistic Schrödinger equation provides a framework for discussion of the continuation of the partial-wave scattering amplitude into complex angular momentum, l. In particular, it has been shown by Regge[1,2] that the partial-wave amplitude continued as a function of complex l, on the basis of the Schrödinger equation, is analytic in the right-half l plane. Poles in the right-half l plane correspond to the resonances and bound states of an attractive potential and are called Regge poles. Recently, several authors[3,4] have realized the importance of Regge poles in strong interactions that are intrinsically relativistic in nature. These poles are important for an understanding of the analytically continued S matrix in energy and momentum transfer and for the formulation of the principles of particle equivalence and maximal strength of strong interactions. It is the high-energy behavior of scattering amplitudes (in particular, the nucleon-nucleon elastic amplitude) that is our primary concern in this paper. If the Regge pole conjecture is accepted, then scattering cross sections at high energies are controlled in a very simple way by poles in "crossed channels."

For a relativistic scattering amplitude, Froissart and Gribov have proposed, on the basis of the Mandelstam representation, a particular continuation of the partial-wave amplitude from physical values of angular momentum (positive integers) into complex angular momentum.[5] Squires and Prosperi give conditions for the uniqueness of this continuation.[6]

All these considerations (with the possible exception of Gell-Mann *et al.*) have been made for spinless particles. This paper has a twofold objective; namely, to verify that the above results have an analog in the two-nucleon problem when spin is taken into account, and to obtain results applicable to the high-energy nucleon-nucleon problem.

Section 2 is a brief discussion of kinematical preliminaries. A complete discussion of the kinematics for the Mandelstam representation is given by Goldberger, Grisaru, MacDowell, and Wong (GGMW)[7]; however, this section is self-contained, and the reader is not expected to be familiar with all the results of GGMW.

In Sec. 3, a unique continuation of the partial-wave helicity amplitudes corresponding to transitions of definite parity is derived from the Mandelstam representation. It is in this section that an alternate set of amplitudes can be defined that simplifies the discussion of analyticity in angular momentum for the partial-wave helicity amplitudes. The Sommerfeld-Watson representation[1,2] for the nucleon-nucleon amplitude is given. The results of this section are applicable to both nucleon-nucleon scattering and nucleon antinucleon scattering.

In both nucleon-nucleon (NN) and nucleon-antinucleon ($N\bar{N}$) scattering there are five independent amplitudes, because of the spin. In Sec. 4, the question is settled as to which linear combinations of the five independent partial-wave $N\bar{N}$ helicity amplitudes are associated with the various Regge poles having the quantum numbers of the $N\bar{N}$ system. The results of this section are presented in Table I.

In Sec. 5, a discussion similar to that in Sec. 4 for Regge poles with the NN quantum numbers. Sections 4 and 5, although qualitative in nature, are included because the results are important for practical calculations.

Section 6 is devoted to the study of high-energy (NN) scattering. Formulas for the high-energy total cross section and angular distribution are derived in terms of the Regge poles in the $N\bar{N}$ channels (crossed channels). The trajectory of a Regge pole determines the high-momentum transfer behavior of the nucleon-antinucleon amplitude and hence (by crossing) the high-energy

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[1] T. Regge, Nuovo Cimento **14**, 951 (1959).
[2] T. Regge, Nuovo Cimento **18**, 947 (1960).
[3] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962); G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).
[4] M. Gell-Mann, F. Zachariasen, and S. C. Frautschi, Phys. Rev. **126**, 2204 (1962); R. Blankenbecler and M. L. Goldberger, *ibid.* **126**, 766 (1962).
[5] M. Froissart, in the *Proceedings of the La Jolla Conference on the Theory of Weak and Strong Interactions;* V. N. Gribov, Zh Eksperim. i Teor. Fiz. **41**, 667, 1962 (1961) [translations: Soviet Phys.—JETP **14**, 478, 1395 (1962)].
[6] Evan J. Squires, Nuovo Cimento **25**, 242 (1962); G. M. Prosperi, *ibid.* **24**, 957 (1962).
[7] M. L. Goldberger, M. F. Grisaru, S. W. MacDowell, and D. Y. Wong (GGMW), Phys. Rev. **120**, 2250 (1960).

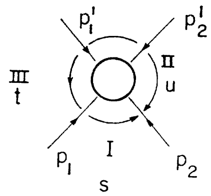

FIG. 1. Scattering diagram for the nucleon-nucleon problem.

behavior of the nucleon-nucleon amplitude is also determined. The contribution to backward np scattering of the π and the ρ trajectories, in particular, is discussed in some detail.

2. KINEMATICS

A complete discussion of the kinematical preliminaries, crossing, and the choice of amplitudes suitable for the Mandelstam representation has been given by GGMW.[7] However, questions relevant to our purposes are discussed in this section.

There are three physical processes related by analytic continuation of the momentum variables,

$$\begin{aligned}&\text{(I)}\quad N_1+N_2\to N_{1'}+N_{2'},\\&\text{(II)}\quad N_1+\bar{N}_{2'}\to N_{1'}+\bar{N}_2,\\ \text{and}\quad&\text{(III)}\quad N_1+\bar{N}_{1'}\to \bar{N}_2+N_{2'},\end{aligned}\tag{2.1}$$

where the bars indicate antinucleons. The four-momenta of the particles 1, 2, 1′, and 2′ are denoted p_1, p_2, $p_{1'}$, and $p_{2'}$, respectively, and all momenta are taken to be into the scattering diagram Fig. 1. Each of the momenta has the property $(p_i)^2=m^2$, where m is the nucleon mass. The metric chosen here is such that $x\cdot y=x_4y_4-\mathbf{x}\cdot\mathbf{y}$, where x and y are four-vectors.

The customary scalar invariants are defined:

$$\begin{aligned}s&=(p_1+p_2)^2=(p_{1'}+p_{2'})^2,\\t&=(p_1+p_{1'})^2=(p_2+p_{2'})^2,\\ \text{and}\quad u&=(p_1+p_{2'})^2=(p_{1'}+p_2)^2,\end{aligned}\tag{2.2}$$

with the constraint,

$$s+t+u=4m^2,\tag{2.3}$$

and conservation of four-momenta reads

$$p_1+p_2+p_{1'}+p_{2'}=0.\tag{2.4}$$

In nucleon-nucleon scattering, particles $N_{1'}$ and $N_{2'}$ are outgoing and their momenta are described by $-p_{1'}$ and $-p_{2'}$; the variables s, t, and u are related to center-of-mass quantities for nucleon-nucleon scattering by

$$\begin{aligned}s&=4E^2=4(p^2+m^2),\\t&=-2p^2(1-z),\\ \text{and}\quad u&=-2p^2(1+z),\end{aligned}\tag{2.5}$$

where p is the absolute value of the three-momenta of either particle and E is the energy of either particle in the center-of-mass system (c.m.). The quantity $z=\cos\theta$ is the cosine of the c.m. scattering angle. The physical region for the invariant variables is

$$s>4m^2,\quad t<0,\quad \text{and}\quad u<0.\tag{2.6}$$

This range of variables is designated as the nucleon-nucleon channel or the "s channel."

For the nucleon-antinucleon process GGMW is followed and we choose p_1 to be the momentum of the incoming nucleon, $p_{2'}$ the momentum of the incoming antinucleon, $-p_{1'}$ the momentum of the outgoing nucleon, and $-p_2$ the momentum of the outgoing antinucleon. In terms of c.m. quantities the invariant variables take the form

$$\begin{aligned}s&=-2p_u^2(1+z_u),\\t&=-2p_u^2(1-z_u),\\ \text{and}\quad u&=4(p_u^2+m^2)=4E_u^2.\end{aligned}\tag{2.7}$$

The physical region of these variables is

$$s<0,\quad t<0,\quad \text{and}\quad u>4m^2.\tag{2.8}$$

This domain of variables is referred to as the "u channel."

There is another nucleon-antinucleon channel, the "t channel." In terms of c.m. quantities the invariant variables take the form

$$\begin{aligned}s&=-2p_t^2(1+z_t),\\t&=4(p_t^2+m^2)=4E_t^2,\\ \text{and}\quad u&=-2p_t^2(1-z_t),\end{aligned}\tag{2.9}$$

where $s<0$, $t>4m^2$, and $u<0$ is the physical region.

In all the following, charge independence is assumed to be valid and the formalism of isotopic spin is used. Space-reflection invariance and time-reversal invariance are assumed to be valid throughout. These symmetries, together with Pauli symmetry, limit the number of amplitudes in nucleon-nucleon scattering to five for each value of total isotopic spin (0,1). Similarly, the above symmetries, together with G parity, limit the number of amplitudes to five for nucleon-antinucleon scattering.

The S matrix for nucleon-nucleon scattering is written

$$\begin{aligned}&\langle\lambda',\mu';\mathbf{p}'|(S-1)|\lambda,\mu;\mathbf{p}\rangle\\&\quad=-i(2\pi)^{-2}(m/E)^2\delta^{(4)}(p_1+p_2+p_{1'}+p_{2'})\mathfrak{T}^T,\end{aligned}\tag{2.10}$$

where $\lambda'\mu'$ and λ, u are the helicities of the final and initial particles, respectively; $(-\mathbf{p}',\mathbf{p}')$ and $(-\mathbf{p},\mathbf{p})$ are the final and initial c.m. momenta. Since conservation of isotopic spin is assumed, the S matrix depends upon the total isotopic spin T, and not the components of T. The normalization of the amplitude $\mathfrak{T}^T$, called the Feynman amplitude, is such that the differential cross section per unit solid angle (c.m.) is

$$d\sigma/d\Omega=|\phi_{\lambda'\mu',\lambda\mu}(\mathbf{p}',\mathbf{p})|^2,\tag{2.11}$$

where

$$\phi_{\lambda'\mu',\lambda\mu}(\mathbf{p}',\mathbf{p})=(m^2/4\pi E)\mathfrak{T}^T.\tag{2.12}$$

It follows directly from rotational invariance that the amplitude ϕ can be developed in terms of the partial-wave helicity amplitudes by the expansion[8]

$$\phi_{\lambda'\mu',\lambda\mu}(\mathbf{p}',\mathbf{p})=\phi_{\lambda'\mu',\lambda\mu}(W,z)$$
$$=\frac{1}{p}\sum_{J=0}^{\infty}(2J+1)d_{\lambda-\mu,\lambda'-\mu'}{}^{J}(\theta)T_{\lambda'\mu',\lambda\mu}{}^{J}(W). \tag{2.13}$$

The beam is incident from the z direction and is scattered into the Euler angles $(0,\theta,0)$. The quantity $T_{\lambda'\mu'\lambda\mu}{}^{J}$ is the partial-wave helicity amplitude and is proportional to the S matrix in the angular momentum representation

$$T_{\lambda'\mu',\lambda\mu}{}^{J}(W)=(1/2i)(S_{\lambda'\mu',\lambda\mu}{}^{J}-\delta_{\lambda'\lambda}\delta_{\mu'\mu}), \tag{2.14}$$

where $W=2E$, and

$$\langle(J',M');\lambda'\mu';p'|S|(J,M);\lambda\mu;p\rangle$$
$$=\delta_{JJ'}\delta_{MM'}\delta(W'-W)S_{\lambda'\mu',\lambda\mu}{}^{J}. \tag{2.15}$$

The functions $d_{mn}{}^{J}(\theta)$ are reduced-rotation matrices, and have simple orthogonality properties[9] which lead to

$$T_{\lambda'\mu',\lambda\mu}{}^{J}(W)=\frac{p}{2}\int_{-1}^{1}dz\, d_{\lambda-\mu,\lambda'-\mu'}{}^{J}(\theta)\phi_{\lambda'\mu',\lambda\mu}(W,z). \tag{2.16}$$

Time-reversal invariance, conservation of total spin (which follows from charge independence and Pauli symmetry), and space-reflection invariance lead to the following symmetries for the partial-wave helicity amplitudes, respectively:

$$T_{\lambda\mu,\lambda'\mu'}{}^{J}(W)=T_{\lambda'\mu',\lambda\mu}{}^{J}(W),$$
$$T_{\mu'\lambda',\mu\lambda}{}^{J}(W)=T_{\lambda'\mu',\lambda\mu}{}^{J}(W), \tag{2.17}$$

and

$$T_{-\lambda'-\mu',-\lambda-\mu}{}^{J}(W)=T_{\lambda'\mu',\lambda\mu}{}^{J}(W).$$

The indices λ, μ, etc., are two-valued $(\pm 1/2)$, and if one counts properly there are 16 configurations of helicities of the initial and final nucleons. However, the symmetries (2.17) reduce this number to five independent helicity amplitudes for each isotopic spin, and, following GGMW, the independent amplitudes are, for each value of the isotopic spin $T=0$, 1:

$$T_1{}^{J,T}(W)=T_{1/2\ 1/2,1/2\ 1/2}{}^{J,T}(W),$$
$$T_2{}^{J,T}(W)=T_{1/2\ 1/2,-1/2\ -1/2}{}^{J,T}(W),$$
$$T_3{}^{J,T}(W)=T_{1/2\ -1/2,1/2\ -1/2}{}^{J,T}(W), \tag{2.18}$$
$$T_4{}^{J,T}(W)=T_{1/2\ -1/2,-1/2\ 1/2}{}^{J,T}(W),$$

and

$$T_5{}^{J,T}(W)=T_{1/2\ 1/2,1/2\ -1/2}{}^{J,T}(W),$$

and likewise for the set $\phi_{\lambda'\mu',\lambda\mu}$.

The partial-wave helicity amplitudes (2.18) can be combined to give the amplitudes for definite total spin $S(S=0,1)$ and definite parity $(-1)^L$, where L is the relative orbital-angular momentum of the two nucleons. The appropriate combinations are

$$\text{Singlet}(J=L)f^{J,T}(W)=T_1{}^{J,T}(W)-T_2{}^{J,T}(W),$$
$$\text{Triplet}(J=L)f^{J,T}(W)=T_3{}^{J,T}(W)-T_4{}^{J,T}(W),$$

and Triplet$(J=L\pm 1)$:

$$f_{11}{}^{J,T}(W)=T_1{}^{J,T}(W)+T_2{}^{J,T}(W), \tag{2.19}$$
$$f_{12}{}^{J,T}(W)=2T_5{}^{J,T}(W),$$
$$f_{22}{}^{J,T}(W)=T_3{}^{J,T}(W)+T_4{}^{J,T}(W).$$

Because the Regge poles that correspond to definite parity are to be considered, the set (2.19) is appropriate for continuation into complex angular momentum.

Also of importance for the next section is a choice of amplitudes suitable for the Mandelstam representation; this problem has been dealt with in detail by GGMW. By use of four-component helicity spinors for the initial and final fermions and expressing $\mathfrak{T}^T$ as an operator in the Dirac spinor space, a set of invariant amplitudes $G_i{}^T(s,u,t)$ is found which satisfy the Mandelstam representation. This set is related to the ϕ's by

$$2E\phi_1=zm^2G_2+E^2G_1+m^2G_3+m^2zG_4-p^2G_5,$$
$$2E\phi_2=-E^2G_1+z(p^2+E^2)G_2-m^2G_3+m^2zG_4-p^2G_5,$$
$$2E\phi_3=(1+z)(-p^2G_3+m^2G_2+E^2G_4), \tag{2.20}$$
$$2E\phi_4=(1-z)(p^2G_3+m^2G_2+E^2G_4),$$

and

$$2m\phi_5=-m^2(1-z^2)^{1/2}(G_2+G_4).$$

The Pauli principle, which requires that the S matrix be odd under interchange of the quantum numbers of the two nucleons in either the initial or final state, assumes the form

$$G_i{}^T(s,u,t)=(-1)^{i+T}G_i{}^T(s,t,u). \tag{2.21}$$

The fixed-s dispersion relation takes the usual form

$$G_i{}^T(s,u,t)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{{}_tD_i{}^T(s,t')dt'}{t'+2p^2(1-z)}$$
$$+\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{{}_uD_i{}^T(s,u')du'}{u'+2p^2(1+z)}, \tag{2.22}$$

where ${}_tD$ and ${}_uD$ are the absorptive parts in the t and u channels, respectively, and m_π is the pion mass. The lower limits of integration follow from physical considerations of the least massive intermediate states with the quantum numbers of the nucleon-antinucleon pair. The one-meson exchange contribution is not displayed.

The discussion in this section could have been carried out for nucleon-antinucleon scattering with a slight modification; namely, that G parity replaces Pauli symmetry. There is an analogous set of amplitudes $\bar{G}_i{}^T(u,s,t)$ for the u channel, related to the set

[8] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[9] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

$G_i^T(s,u,t)$ by the crossing matrix (GGMW):

$$G_i^T(s,u,t)=\sum_{jT'} \Delta_{ij}B^{TT'}\bar{G}_j^{T'}(u,s,t), \tag{2.23}$$

where

$$B=\frac{1}{2}\begin{bmatrix}-1 & 3\\ 1 & 1\end{bmatrix}\text{(isotopic-spin crossing matrix)} \tag{2.24}$$

and

$$\Delta=\tfrac{1}{4}\begin{bmatrix}-1 & 6 & 4 & -4 & -1\\ 1 & 2 & 0 & 0 & 1\\ 1 & 0 & 2 & 2 & -1\\ -1 & 0 & 2 & 2 & 1\\ -1 & 6 & -4 & 4 & -1\end{bmatrix}. \tag{2.24'}$$

There is also a set of amplitudes $\bar{\phi}$ for the u channel; the amplitudes are related to the $\bar{G}$'s by an equation of the form (2.20) if p, E, and z are replaced by p_u, E_u, and z_u.

3. ANALYTIC CONTINUATION INTO COMPLEX J

In this section it is shown that, given the set of amplitudes (2.19) defined at the physical values of J by Eq. (2.16), the fixed-energy dispersion relation (2.22), and the assumption that the absorptive parts $_tD$ and $_uD$ behave like powers of t and u as $(t,u)\to\infty$, it is possible to define a set of functions $f(J,W)$ with the following properties: (i) $f(J,W)=f^J(W)$ for $J=0, 1, 2, \cdots$ (physical J) (Eq. 2.19); (ii) $f(J,W)$ is holomorphic for $\mathrm{Re}J>N(s)$ in the complex J plane; and (iii) $|f(J,W)|<\exp-\tau\,\mathrm{Re}J$, $\tau>0$, uniformly for J sufficiently large. The indices on the partial-wave amplitudes $f^J(W)$ have been omitted here. A function of complex J with the properties (i), (ii), and (iii) is unique; the proof of the uniqueness is given by Prosperi.[6]

Furthermore, it is assumed, following the work of Chew and Frautschi,[3] that it is possible to move the boundary N ($\mathrm{Re}J>N$) to the left in the complex J plane and that only poles will appear. This assumption puts the relativistic problem on the same footing as the nonrelativistic problem for superpositions of Yukawa potentials, where Regge[1,2] has given a continuation with properties (i) and (iii) but (ii) is replaced by meromorphy for $\mathrm{Re}J>-1/2$.

In attempting to find a set of functions with properties (i), (ii), and (iii), it is found that even and odd J have to be treated separately; this leads to the concept of J parity which has been introduced for spinless particles by several authors.[4,6]

For a set of functions with property (iii), it is possible to construct the Sommerfeld-Watson representation for the amplitudes ϕ or G, and this section concludes with a discussion of the Sommerfeld-Watson representation.

To remove certain trivial nonanalytic factors, we consider, instead, the set of functions defined by

$$\begin{aligned}h_0^J&=(E/p)f_0^J,\\ h_{11}^J&=(E/p)f_{11}^J,\\ h_{22}^J&=(E/p)f_{22}^J,\\ h_1^J&=(E/p)f_1^J,\end{aligned} \tag{3.1}$$

and

$$h_{12}^J=(m/p)f_{12}^J.$$

By use of Eqs. (3.1), (2.18), and (2.16), the set is related to the set ϕ by

$$h_0^J(s)=\frac{1}{2}\int_{-1}^{1}dz\,[E\phi_1(W,z)-E\phi_2(W,z)]d_{00}^J(\theta),$$

$$h_{11}^J(s)=\frac{1}{2}\int_{-1}^{1}dz\,[E\phi_1(W,z)+E\phi_2(W,z)]d_{00}^J(\theta),$$

$$h_{22}^J(s)=\frac{1}{2}\int_{-1}^{1}dz\,[E\phi_3(W,z)d_{11}^J(\theta)+E\phi_4(W,z)d_{-11}^J(\theta)], \tag{3.2}$$

$$h_1^J(s)=\frac{1}{2}\int_{-1}^{1}dz\,[E\phi_3(W,z)d_{11}^J(\theta)-E\phi_4(W,z)d_{-11}^J(\theta)],$$

and

$$h_{12}^J(s)=\frac{1}{2}\int_{-1}^{1}dz\,m\phi_5(W,z)d_{10}^J(\theta).$$

The h's are even functions of W and E and, hence, functions only of $s=W^2$ because $E\phi_i(W,z)$, $1(i)4$, and $m\phi_5(W,z)$ are related to the set G through relations of the type (2.20), and the G's are functions of s, t, and u. Next we use Eq. (2.20) and notice that the angle-dependent factors are

$$\begin{aligned}z&=d_{00}^1(\theta),\\ \tfrac{1}{2}(1+z)&=d_{11}^1(\theta)=d_{-1-1}^1(\theta),\\ \tfrac{1}{2}(1-z)&=d_{1-1}^1(\theta)=d_{-11}^1(\theta),\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}(1-z^2)^{1/2}&=-\sqrt{2}d_{10}^1(\theta)=-\sqrt{2}d_{0-1}^1(\theta)\\ &=\sqrt{2}d_{-10}^1(\theta)=\sqrt{2}d_{01}^1(\theta).\end{aligned}$$

When Eq. (2.20) is substituted into Eq. (3.2), a product of d^J functions is encountered because of the effect of Eq. (3.3), and, to get rid of the d^J functions entirely, we use the identity

$$d_{mn}^J(\theta)d_{-m-n}^1(\theta)=\sum_{k=J-1}^{k=J+1}C(J,1,k;m,-m,0)\times C(J,1,k;n,-n,0)P_k(\cos\theta). \tag{3.4}$$

The foregoing identity follows directly from Eq. (4.25) of Rose[9] and the fact that $d_{00}^k(\theta)=P_k(\cos\theta)$. The quantities $C(J,1,k;m,-m,0)$ are Clebsch-Gordan coefficients and their phases are fixed by the conventions of Rose.[9] The following set of equations results:

$$h_0^{J,T}(s)=E^2g_1^{J,T}(s)-p^2\left[\frac{Jg_2^{J-1,T}(s)+(J+1)g_2^{J+1,T}(s)}{2J+1}\right]+m^2g_3^{J,T}(s),$$

$$h_{11}^{J,T}(s)=E^2\left[\frac{Jg_2^{J-1,T}(s)+(J+1)g_2^{J+1,T}(s)}{2J+1}\right]+m^2\left[\frac{Jg_4^{J-1,T}(s)+(J+1)g_4^{J+1,T}(s)}{2J+1}\right]-p^2g_5^{J,T}(s),$$

$$h_{22}^{J,T}(s)=-p^2g_3^{J,T}(s)+m^2\left[\frac{(J+1)g_2^{J-1,T}(s)+Jg_2^{J+1,T}(s)}{2J+1}\right]+E^2\left[\frac{(J+1)g_4^{J-1,T}(s)+Jg_4^{J+1,T}(s)}{2J+1}\right], \quad (3.5)$$

$$h_1^{J,T}(s)=-p^2\left[\frac{(J+1)g_3^{J-1,T}(s)+Jg_3^{J+1,T}(s)}{2J+1}\right]+m^2g_2^{J,T}(s)+E^2g_4^{J,T}(s),$$

and

$$h_{12}^{J,T}(s)=\frac{m^2}{2}\frac{[J(J+1)]^{1/2}}{2J+1}[g_2^{J+1,T}(s)-g_2^{J-1,T}(s)+g_4^{J+1,T}(s)-g_4^{J-1,T}(s)],$$

where

$$g_i^{k,T}=\frac{1}{2}\int_{-1}^{1}P_k(z)G_i^T(s,u,t)dz. \quad (3.6)$$

The fixed-energy dispersion relation (2.22) for $G_i^T(s,u,t)$ is substituted into Eq. (3.6), and we obtain

$$g_i^{k,T}(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}{}_tD_i^T(s,t')\frac{1}{2}\int_{-1}^{1}\frac{dz\,P_k(z)}{t'+2p^2(1-z)}dt'+\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}{}_uD_i^T(s,t')\frac{1}{2}\int_{-1}^{1}\frac{dz\,P_k(z)}{t'+2p^2(1+z)}dt'. \quad (3.7)$$

The order of integration has been interchanged [which is certainly permissible if the integrals in Eq. (2.22) exist uniformly in z]. The dispersion relation was written down without subtractions and, to be completely rigorous, the subtractions should be included; however, the explicit display of the subtractions only complicates the algebra.

Using the Neumann representation for the Legendre function of the second kind (Whittaker and Watson),[10]

$$Q_k(z)=\frac{1}{2}\int_{-1}^{1}\frac{P_k(x)dx}{z-x}, \quad k=0,1,2,\cdots, \quad (3.8)$$

we have

$$g_i^{k,T}(s)=\frac{1}{2\pi p^2}\int_{4m_\pi^2}^{\infty}Q_k\left(1+\frac{t'}{2p^2}\right)\times[{}_tD_i^T(s,t')+(-)^k\,{}_uD_i^T(s,t')]dt'. \quad (3.9)$$

If we use Eq. (3.9) in Eq. (3.5) and analytically continue in J, the resulting set of functions $h(J,s)$ will certainly satisfy condition (i) since all the steps in arriving at Eq. (3.9) were true for integral J.

Next we use the assumption that $D_i(s,t)$ behaves like a power of t as t approaches infinity,

$$D_i(s,t)\lesssim t^{N(s)}, \quad (3.10)$$

where $N(s)$ is, in general, some complex-valued function of s. With this assumption and with knowledge of the asymptotic behavior of $Q_k(z)$ for large z [i.e., $Q_k(z)\sim 1/z^{k+1}$], we see that the integral (3.9) converges and defines a holomorphic function of k in the region[5]

$$\mathrm{Re}k>N(s). \quad (3.11)$$

The only remaining problem is to establish property (iii); the function $Q_k(z)$ has the integral representation[11]

$$Q_k(z)=\int_{z+(z^2-1)^{1/2}}^{\infty}\xi^{-k-1}(1-2\xi z+\xi^2)^{-1/2}d\xi, \quad \mathrm{Re}k>-1. \quad (3.12)$$

Changing the variable of integration in (3.9) to $z=1+t/2p^2$, using (3.12), and interchanging the order of integration, we have

$$g_i^T(k,s)=\frac{1}{\pi}\int_{z_0+(z_0^2-1)^{1/2}}^{\infty}d\xi\frac{{}_t\sigma_i(s,\xi)+(-1)^k\,{}_u\sigma_i(s,\xi)}{\xi^{k+1}}, \quad (3.13)$$

where

$${}_{t,u}\sigma_i(s,\xi)=\int_{z_0}^{\frac{1}{2}(\xi+1/\xi)}\frac{{}_{t,u}D_i(s,z)dz}{(1-2\xi z+\xi^2)^{1/2}},$$

and

$$z_0=1+2(m_\pi)^2/p^2.$$

If it is assumed that $D(s,z)$ behaves like a power of z at infinity, then it follows that $\sigma(s,\xi)$ behaves like the same power as ξ approaches infinity, and we write, for $\sigma(s,\xi)$,

$$\sigma(s,\xi)\lesssim\xi^N \quad \text{as} \quad \xi\to\infty. \quad (3.15)$$

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[10] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1927).

[11] This argument is similar to that of M. Froissart, Phys. Rev. **123**, 1053 (1961).

Thus, the integrals $\int[{}_t\sigma_i(s,\xi)/\xi^{k+1}]$ and $\int[{}_u\sigma_i(s,\xi)/\xi^{k+1}]$ are bounded:

$$\left|\int_{z_0+(z_0^2-1)^{1/2}}^{\infty}\frac{{}_{t,u}\sigma_i(s,\xi)}{\xi^{k+1}}d\xi\right|\leq\int_{z_0+(z_0^2-1)^{1/2}}^{\infty}\xi^N\xi^{-k-1}d\xi$$
$$=\frac{1}{k-N}\left[\frac{1}{z_0+(z_0^2-1)^{1/2}}\right]^{k-N}=\frac{1}{k-N}\exp(N-k)$$
$$\times\ln[z_0+(z_0^2-1)^{1/2}],\quad(3.16)$$

since $\ln[z_0+(z_0^2-1)^{1/2}]>0$ for $p^2>0$. However, the factor $(-1)^k$ spoils the desired exponential behavior for large k; because of this, two different sets of functions are defined,

$$g_i^{T(\pm)}(k,s)=\frac{1}{2\pi p^2}\int_{4(m_\pi)^2}^{\infty}Q_k\left(1+\frac{t'}{2p^2}\right)$$
$$\times[{}_tD_i^T(s,t)\pm{}_uD_i^T(s,t')]dt',\quad(3.17)$$

with the properties

$$g_i^{T(+)}(k,s)=g_i^{k,T}(s),\quad\text{where}\quad k=0,2,4,\cdots,$$

and (3.18)

$$g_i^{T(-)}(k,s)=g_i^{k,T}(s),\quad\text{where}\quad k=1,3,5,\cdots.$$

The functions $g_i^{T(\pm)}(k,s)$ each have the desired exponential decrease for large k. We can now find the even and odd J-parity amplitudes $h^{(\pm)}$ by evaluating the Clebsch-Gordan coefficients and separating even and odd J in Eq. (3.5). (The physical significance of J parity is discussed in Sec. 4.) We obtain a set of equations identical to Eq. (3.5), except for the replacements

$$h_i^{J,T}(s)\quad\text{for}\quad J=0,2,4,\cdots\rightarrow h_i^{(+),T}(J,s)\quad\text{where}\quad i=0,11,1,22,12,$$
$$h_i^{J,T}(s)\quad\text{for}\quad J=1,3,5,\cdots\rightarrow h_i^{(-)T}(J,s)\quad\text{where}\quad i=0,11,1,22,12;$$

and (3.19)

$$g_i^{k,T}(s)\quad\text{for}\quad k=0,2,4,\rightarrow g_i^{(+)T}(k,s)\quad\text{where}\quad i=1,2,3,4,5,$$
$$g_i^{k,T}(s)\quad\text{for}\quad k=1,3,5,\rightarrow g_i^{(-)}(k,s)\quad\text{where}\quad i=1,2,3,4,5.$$

Equation (3.5), with the above replacements, satisfies properties (i), (ii), and (iii). The coefficients in front of the g's which depend upon J do not change the asymptotic behavior in J so that condition (iii) is satisfied and the set is unique.

If one expects an extension to the left-half plane with Eqs. (3.19) and (3.17), the factor of $2J+1$ in the denominator of certain terms in Eq. (3.19) is disturbing, but does not lead to a fixed pole in J. A typical such term is

$$\frac{(J+1)g_2^{(\mp)T}(J-1,s)+Jg_2^{(\mp)T}(J+1,s)}{2J+1}$$
$$=\frac{1}{2\pi p^2}\frac{1}{2J+1}\int_{4m_\pi^2}^{\infty}\left[(J+1)Q_{J-1}\left(1+\frac{t'}{2p^2}\right)\right.$$
$$\left.+JQ_{J+1}\left(1+\frac{t'}{2p^2}\right)\right][{}_tD_2^T(s,t')\mp{}_uD^T(s,t')]dt';$$
$$(3.20)$$

by use of the property of the Q functions for n=half-integer $Q_n(z)=Q_{-n-1}(z)$, the numerator in Eq. (3.20) goes to zero when $2J+1$ goes to zero. Thus, the factors of $(2J+1)$ do not cause singularities in J. The only other disturbing factor is $[J(J+1)]^{1/2}$ in h_{12} of Eq. (3.19); but, instead of continuing $h_{12}^{(\pm)T}(J,s)$, we may consider

$$h_{12}^{(\pm)T}(J,s)/[J(J+1)]^{1/2},\quad(3.21)$$

this quantity being related to the ϕ_5 scattering amplitude through the Sommerfeld-Watson transformation.

Following the work of Chew and Frautschi,[3] it is assumed that one can move to the left of the boundary N (3–11) and only Regge poles will appear. The position α of a particular Regge pole depends upon the energy s. In the nonrelativistic problem for superpositions of Yukawa potentials, it has been shown by Regge[1,2] that the only limitation on the region of meromorphy is the line $\mathrm{Re}J=-1/2$. Also, it has been shown by Froissart,[12] Regge,[12] and Mandelstam[13] that the boundary of meromorphy can be moved arbitrarily far to the left (left-hand J plane) for the nonrelativistic problem. In the relativistic problem, Froissart has shown that the boundary of holomorphy can be extended to $\mathrm{Re}J>1$ for negative s.[11] Also, several authors have shown that it is possible to prove meromorphy for $\mathrm{Re}J>1$ in the relativistic problem.[14]

A. Sommerfeld-Watson Transformation

Using the partial-wave expansion (2.13) and the relations between the partial-wave helicity amplitudes and the set $h(s)$, Eqs. (2.19) and (3.1), we obtain the following set of expansions:

$$E[\phi_1(W,z)-\phi_2(W,z)]$$
$$=\sum_{J\text{ even}}h_0^{(+)J}(s)P_J(z)(2J+1)$$
$$+\sum_{J\text{ odd}}h_0^{(-)J}(s)P_J(z)(2J+1),$$

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[12] M. Froissart, J. Math. Phys. **3**, 922 (1962); T. Regge, Nuovo Cimento **24**, 518 (1962).
[13] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).
[14] K. Bardacki, Phys. Rev. **127**, 1832 (1962); A. O. Barut and D. Zwanziger, *ibid.* **127**, 974 (1962); G. M. Prosperi, Nuovo Cimento **26**, 541 (1962).

$$E[\phi_1(W,z)+\phi_2(W,z)]$$
$$=\sum_{J\text{ even}} h_{11}{}^{(+)J}(s)P_J(z)(2J+1)$$
$$+\sum_{J\text{ odd}} h_{11}{}^{(-)J}(s)P_J(Z)(2J+1),$$
$$E\phi_3(W,z)$$
$$=\tfrac{1}{2}\sum_{J\text{ even}}[h_1{}^{(+)J}(s)+h_{22}{}^{(+)J}(s)]d_{11}{}^J(\theta)(2J+1)$$
$$+\tfrac{1}{2}\sum_{J\text{ odd}}[h_1{}^{(-)J}(s)+h_{22}{}^{(-)J}(s)]d_{11}{}^J(\theta)(2J+1),$$
$$E\phi_4(W,z)$$
$$=\tfrac{1}{2}\sum_{J\text{ even}}[h_{22}{}^{(+)J}(s)-h_1{}^{(+)J}(s)]d_{-11}{}^J(\theta)(2J+1)$$
$$+\tfrac{1}{2}\sum_{J\text{ odd}}[h_{22}{}^{(-)J}(s)-h_1{}^{(-)J}(s)]d_{-11}{}^J(\theta)(2J+1),$$

and

$$m\phi_5(W,z)=\sum_{J\text{ even}} h_{12}{}^{(+)J}(s)d_{10}{}^J(\theta)(2J+1)$$
$$+\sum_{J\text{ odd}} h_{12}{}^{(-)J}(s)d_{10}{}^J(\theta)(2J+1). \quad (3.22)$$

We can now use the assumption that the set of functions $h^{(\pm)T}(J,s)$ contains only Regge poles, and we can perform the Sommerfeld-Watson transformation and obtain the set of scattering amplitudes $\phi(W,z)$ in terms of the Regge poles. The d^J functions do not cause any difficulties when we perform the Sommerfeld-Watson transformation; however, the details are discussed only for the first of the expansions in Eq. (3.22). The expansions in Eq. (3.22) can be formally written as contour integrals where the contour C encloses all the positive integers in the J plane (Fig. 2):

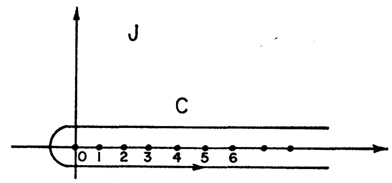

FIG. 2. Contour C in the J plane.

$$E[\phi_1(W,z)-\phi_2(W,z)]$$
$$=\frac{1}{4i}\int_C \frac{dJ}{\sin\pi J}h_0{}^{(+)}(J,s)$$
$$\times[P_J(-z)+P_J(z)](2J+1)+(+\to-) \quad (3.23)$$

[where the symbol $(+\to-)$ indicates that another term is to be added in which $h^{(+)}(J,s)$ is replaced by $h^{(-)}(J,s)$ and $P_J(z)$ is replaced by $-P_J(z)$, and similarly for $\beta^{(+)}$ in (3.24)].

We distort the contour to run along a line $\mathrm{Re}J=N$ parallel to the imaginary axis in the J plane and close it by a large semicircle R (half-plane), where this new contour encloses the poles of $h(J,s)$. The quantity $E[\phi_1-\phi_2]$ then takes the form:

$$E[\phi_1(W,z)-\phi_2(W,z)]=\frac{1}{4i}\int_R \frac{dJ}{\sin\pi J}h_0{}^{(+)}(J,s)[P_J(-z)+P_J(z)](2J+1)$$
$$+\frac{1}{4i}\int_{N-i\infty}^{N+i\infty}\frac{dJ}{\sin\pi J}h_0{}^{(+)}(J,s)[P_J(-z)+P_J(z)](2J+1)$$
$$-\frac{\pi}{2}\sum_{\mathrm{Re}\alpha(n)>N}[2\alpha(n)+1]\frac{\beta_{0,n}{}^{(+)}(s)}{\sin\pi\alpha(n)}[P_{\alpha(n)}(-z)+P_{\alpha(n)}(z)]+(+\to-), \quad (3.24)$$

where $\beta_{0,n}{}^{(\pm)}(s)$ is the residue of the nth pole of $h_0{}^{(\pm)}(J,s)$ at $J=\alpha(n,s)$; in general,

$$\beta_i{}^{(\pm)T}(s)=\lim_{J=\alpha(n,s)}[J-\alpha(n,s)]h_i{}^{(\pm)T}(J,s), \quad (3.25)$$

where $i=0, 1, 11, 22, 12$, and $T=0, 1$. Using the following formulas[15] for the asymptotic behavior of $P_J(\pm z)$ in J:

$$P_J(-z)/\sin\pi J$$
$$\sim\exp(-\mathrm{Re}\theta|\mathrm{Im}J|)\exp(\mathrm{Re}J|\mathrm{Im}\theta|), \quad (3.26)$$
$$P_J(z)/\sin\pi J$$
$$\sim\exp[-(\pi-\mathrm{Re}\theta)|\mathrm{Im}J|]\exp(\mathrm{Re}J|\mathrm{Im}\theta|),$$

and the exponential decrease of $h_0{}^{(\pm)}(J,s)$ for large J derived previously Eqs. (3.16) and (3.19), one sees that the integral over the large circle tends to zero if

$$\mathrm{Im}\theta<\ln[z_0+(z_0{}^2-1)^{1/2}]. \quad (3.27)$$

Condition (3.27) is the Lehmann ellipse,[1,2] which is nothing other than the region in θ for which the partial-wave expansion (3.22) converges. Both the integrals along the line $\mathrm{Re}J=N$ converge if

$$0<\mathrm{Re}\theta<\pi. \quad (3.28)$$

The Regge-pole terms dominate asymptotically over the contribution from the integration parallel to the imaginary axis. We assume that the J plane is free from singularities, except for poles, and that the integration along the line $\mathrm{Re}J=N$ is always to the left of the Regge-pole terms. The extension to the left-half J plane has been considered by Mandelstam.[13]

[15] This asymptotic behavior is due to E. Squires (reference 6).

In a similar way we can perform the Sommerfeld-Watson transformation over the partial-wave helicity amplitudes and obtain the G's directly: By use of the inverse of relation (2.20),[16]

$$G_1=\frac{1}{E}\bigg[(\phi_1-\phi_2)+\frac{m^2}{p^2}\bigg(\frac{\phi_3}{1+z}-\frac{\phi_4}{1-z}\bigg)-z\bigg(\frac{\phi_3}{1+z}+\frac{\phi_4}{1-z}\bigg)-\frac{2z}{(1-z^2)^{1,2}}\frac{E}{m}\phi_5\bigg],$$

$$G_2=-\frac{E}{p^2}\bigg[\frac{\phi_3}{1+z}+\frac{\phi_4}{1-z}+\frac{2E}{m}\frac{\phi_5}{(1-z^2)^{1/2}}\bigg],$$

$$G_3=-\frac{E}{p^2}\bigg[\frac{\phi_3}{1+z}-\frac{\phi_4}{1-z}\bigg],$$

$$G_4=\frac{E}{p^2}\bigg[\frac{\phi_3}{1+z}+\frac{\phi_4}{1-z}\bigg]+\frac{m}{p^2}\frac{2\phi_5}{(1-z^2)^{1/2}},$$

$$G_5=-\frac{E}{p^2}\bigg[\phi_1+\phi_2+z\bigg(\frac{\phi_3}{1+z}+\frac{\phi_4}{1-z}\bigg)\bigg]-\frac{2z}{p^2}\frac{E^2+m^2}{m}\frac{\phi_5}{(1-z^2)^{1/2}},\qquad(3.29)$$

and by use of Eq. (3.22), and the relation of the d^J functions to Legendre functions,

$$d_{11}{}^J(\theta)=\frac{1+z}{J(J+1)}[-(1-z)P_J''(z)+P_J'(z)],$$
$$d_{1-1}{}^J(\theta)=\frac{1-z}{J(J+1)}[(1+z)P_J''(z)+P_J'(z)],\qquad(3.30)$$
$$d_{10}{}^J(\theta)=-\frac{(1-z^2)^{1/2}}{[J(J+1)]^{1/2}}P_J'(z),$$

we have the following set of expansions for the invariant amplitudes:

$$G_1{}^T=\frac{1}{E^2}\sum_{J\text{ even}}(2J+1)\bigg(h_0{}^{(+)J,T}(s)P_J(z)-\frac{h_{22}{}^{(+)J,T}(s)}{J(J+1)}\bigg\{\frac{m^2}{p^2}P_J''(z)+z[zP_J'(z)]'\bigg\}+\frac{h_1{}^{(+)J,T}(s)}{J(J+1)}\bigg\{\frac{m^2}{p^2}[zP_J'(z)]'+zP_J''(z)\bigg\}+\frac{2E^2}{m^2}\frac{h_{12}{}^{(+)J,T}(s)}{[J(J+1)]^{1/2}}zP_J'(z)\bigg)+(\sum_{J\text{ odd}}),$$

$$G_2{}^T=\frac{1}{p^2}\sum_{J\text{ even}}(2J+1)\bigg\{\frac{h_1{}^{(+)J,T}(s)}{J(J+1)}P_J''(z)-\frac{h_{22}{}^{(+)J,T}(s)}{J(J+1)}[zP_J'(z)]'+\frac{2E^2}{m^2}\frac{h_{12}{}^{(+)J,T}(s)}{[J(J+1)]^{1/2}}P_J'(z)\bigg\}+(\sum_{J\text{ odd}}),$$

$$G_3{}^T=\frac{1}{p^2}\sum_{J\text{ even}}\frac{(2J+1)}{J(J+1)}\{h_{22}{}^{(+)J,T}(s)P_J''(z)-h_1{}^{(+)J,T}(s)[zP_J'(z)]'\}+(\sum_{J\text{ odd}}),\qquad(3.31)$$

$$G_4{}^T=\frac{1}{p^2}\sum_{J\text{ even}}\bigg\{\frac{h_{22}{}^{(+)J,T}(s)}{J(J+1)}[zP_J'(z)]'-\frac{h_1{}^{(+)J,T}(s)}{J(J+1)}P_J''(z)-\frac{h_{12}{}^{(+)J,T}(s)}{[J(J+1)]^{1/2}}P_J'(z)\bigg\}+(\sum_{J\text{ odd}}),$$

and

$$G_5{}^T=-\frac{1}{p^2}\sum_{J\text{ even}}(2J+1)\bigg\{h_{11}{}^{(+)J,T}(s)P_J(z)+\frac{h_{22}{}^{(+)J,T}(s)}{J(J+1)}z[zP_J'(z)]'-\frac{h_1{}^{(+)J,T}(s)}{J(J+1)}zP_J''(z)-\frac{2(E^2+m^2)}{m^2}\frac{h_{12}{}^{(+)J,T}(s)}{[J(J+1)]^{1/2}}zP_J'(z)\bigg\}+(\sum_{J\text{ odd}}).$$

The Sommerfeld-Watson transformation can be performed with the expansion (3.31); the presence of derivatives of Legendre functions does not alter the convergence of the integration along the line parallel to the imaginary axis and does not alter the vanishing of the integration around the large semicircle R. We obtain a set of equations similar to Eq. (3.24), except that $h_i{}^{(\pm)T}(J,s)$ in Eq. (3.31) is replaced by $-(\pi/2)\beta_i{}^{(\pm)T}(s)$, $[zP_J'(z)]'$ is replaced by $[zP_\alpha'(z)]'\mp[-zP_\alpha'(-z)]$, and $P_J''(z)$ is replaced by $P_\alpha''(z)\pm P_\alpha''(-z)$.

Asymptotically in z, the quantities $P_\alpha(z)$ and $P_\alpha'(z)$ go into:

$$P_\alpha(z)\sim\frac{\Gamma(\alpha+1/2)2^\alpha z^\alpha}{\Gamma(\alpha+1)\Gamma(1/2)},$$
$$\frac{P_\alpha'(z)}{\alpha(\alpha+1)}\sim\frac{\Gamma(\alpha+3/2)2^{\alpha+1}z^{\alpha-1}}{\Gamma(\alpha+2)\Gamma(1/2)(2\alpha+1)}\quad\text{for}\quad\text{Re}\alpha>-\tfrac{1}{2},\qquad(3.32)$$

[16] The appearance of poles at $J=0$ and $J=-1$ in Eq. (3.29) is spurious, since the numerators also vanish at $J=0$ and $J=-1$, i.e., $P_0''(z)=P_0'(z)=P_{-1}''(z)=P_{-1}'(z)=0$.

and keeping only the leading terms in z, we obtain[17]

$$G_1{}^T\sim-\frac{\pi}{2}\sum_n\frac{1}{\sin\pi\alpha(n)}\frac{1}{E^2}\left\{[2\alpha(n)+1]\beta_{0,n}{}'^{(+)T}(s)-\beta_{22,n}{}'^{(+)T}(s)\alpha(n)+\frac{2E^2}{m^2}\beta_{12,n}{}'(s)\right\}z^{\alpha(n)}\times[1+\exp-i\pi\alpha(n)]+(+\rightarrow-),$$

$$G_2{}^T\sim-\frac{\pi}{2}\sum_n\frac{1}{\sin\pi\alpha(n)}\frac{1}{p^2}\left[-\beta_{22,n}{}'^{(+)T}(s)\alpha(n)+\frac{2E^2}{m^2}\beta_{12,n}{}'^{(+)T}(s)\right]z^{\alpha(n)-1}\{1+\exp-i\pi\alpha(n)\}+(+\rightarrow-),$$

$$G_3{}^T\sim\frac{\pi}{2}\sum_n\frac{1}{\sin\pi\alpha(n)}\frac{1}{p^2}\beta_{1,n}{}'^{(+)T}(s)z^{\alpha(n)-1}\{1+\exp-i\pi\alpha(n)\}+(+\rightarrow-)\tag{3.33}$$

$$G_4{}^T\sim\frac{\pi}{2}\sum_n\frac{1}{\sin\pi\alpha(n)}\frac{1}{p^2}[-\beta_{22,n}{}'^{(+)T}(s)\alpha(n)+2\beta_{12,n}{}'^{(+)T}(s)]z^{\alpha(n)-1}\{1+\exp-i\pi\alpha(n)\}+(+\rightarrow-),$$

$$G_5{}^T\sim\frac{\pi}{2}\sum_n\frac{1}{\sin\pi\alpha(n)}\frac{1}{p^2}\left\{\beta_{11,n}{}'^{(+)T}(s)[2\alpha(n)+1]+\beta_{22,n}{}'^{(+)T}(s)\alpha(n)-\frac{2(E^2+m^2)}{m^2}\beta_{12,n}{}'^{(+)T}(s)\right\}\times z^{\alpha(n)}[1+\exp-i\pi\alpha(n)]+(+\rightarrow-).$$

The foregoing analysis could have been carried out for the $N\bar{N}$ channel. The quantities G, z, p, and E would be replaced by $\bar{G}$, z_u, p_u, and E_u.

4. REGGE POLES IN THE NUCLEON-ANTINUCLEON CHANNEL

In this section the Regge poles with the quantum numbers of the nucleon-antinucleon channel are studied. As stated in the previous section, the position of a Regge pole moves with energy and is said to have a definite trajectory in energy which is controlled by dynamics. Chew and Frautschi have proposed that stable and metastable particles (resonances) are points on Regge trajectories.[3] Each Regge trajectory has a definite set of quantum numbers: isotopic spin, baryon number, strangeness, G parity (if applicable), etc. A particular Regge pole will appear in all S-matrix elements with the quantum numbers in question, regardless of the number and configuration of external particles.

The position of a Regge pole $\alpha(u)$, where u is the square of the center-of-mass energy of the channel in question is conjectured to be an analytic function of u; its imaginary part vanishes for (real) u below the threshold of this channel. The physical points of the Regge trajectory are those points such that $\mathrm{Re}\alpha(u)$ is an integer, and these points correspond to resonances if u is above threshold, and to bound states if u is below threshold. Also, for u above threshold the imaginary part of $\alpha(u)$ at resonance energy is related to the half-width of the unstable configuration by

$$M\Gamma=\mathrm{Im}\alpha/(d\,\mathrm{Re}\alpha/du),\tag{4.1}$$

where M is the mass of the unstable configuration. The real part of $\alpha(u)$ is assumed to be a monotonically increasing function of u for u below threshold and in the region in which resonances occur. Also, $\mathrm{Im}\alpha(u)$ is assumed to be small for sharp resonances. The value of u for which $\alpha(u)$ turns around and acquires a negative slope is purely a question of dynamics.

Also, each Regge trajectory is assumed to have a definite J parity, $(-1)^J$, and the physical points of the Regge trajectory occur for either even or odd J, but not both simultaneously.

The foregoing has been elaborated in detail by Chew and Frautschi.[3] From here on, the number of Regge trajectories accessible to the nucleon-antinucleon channel is studied; in particular, the important question concerning which Regge trajectories are contained in the various amplitudes in Eq. (2.19) is answered.

The quantum numbers of the nucleon-antinucleon systems are as follows:

(a) baryon number = strangeness = 0,
(b) isotopic spin = T = 0, 1,
(c) total spin = S = 0, 1,
(d) parity = P = $(-1)^{L+1}$,
(e) G parity = G = $(-1)^{L+S+T}$,

where L is the relative orbital angular momentum of the nucleon-antinucleon system. Each Regge trajectory is assumed to have definite baryon number, strangeness, isotopic spin, parity, G parity, and J parity. The nucleon-antinucleon system that is coupled to a given

[17] In Eq. (3.33) the residues have been redefined

$$\beta_{0,11}{}'^{(\pm)T}(s)=\frac{\Gamma(\alpha+1/2)2^\alpha}{\Gamma(\alpha+1)\Gamma(1/2)}\beta_{0,11}{}^{(\pm)T}(s),$$

$$\beta_{1,22}{}'^{(\pm)T}(s)=\frac{\Gamma(\alpha+3/2)2^{\alpha+1}}{\Gamma(\alpha+2)\Gamma(1/2)}\beta_{1,22}{}^{(\pm)T}(s),$$

and

$$\beta_{12}{}'^{(\pm)T}(s)=\frac{\Gamma(\alpha+3/2)2^{\alpha+1}}{[\alpha(\alpha+1)]^{1/2}\Gamma(\alpha+2)\Gamma(1/2)}\beta_{12}{}^{(\pm)T}(s).$$

The quantities $\tilde{\beta}_i{}^{(\pm)T}(u)$ are defined as follows $\tilde{\beta}_i{}^{(\pm)T}(u)=\beta_i{}'^{(\pm)T}(u)(s_0/2p_u{}^2)^\alpha$, where $i=0, 1, 11, 12, 22$ and s_0 has been taken to be $2m^2$ by several authors (reference 4).

Regge trajectory has the same quantum numbers as this trajectory; thus, (a), (b), (d), and (e) are specified. The total spin is not independent but is specified by

$$(-1)^S = -P(-1)^T G, \tag{4.3}$$

and the right-hand side of Eq. (4.3) is specified by the Regge trajectory. Since $(-1)^S$ is given by Eq. (4.3) and S is either 0 or 1 for two spin-1/2 particles, S is determined. Therefore, if the quantum numbers of a given Regge trajectory that is coupled to the nucleon-antinucleon channel are specified, then this trajectory is contained in either the singlet or triplet partial-wave amplitudes (2.19), but not both.

There are two important classes of trajectories to be considered separately for the nucleon-antinucleon channel; they are

$$\text{(a)}\quad P(-1)^T G = -1,$$
and
$$\text{(b)}\quad P(-1)^T G = 1. \tag{4.4}$$

It is easily seen from Eq. (4.3) that S is 0 for class (a) and 1 for class (b).

The J parity of a given trajectory is determined by the even or odd nature of $(-1)^J$. For trajectories of class (a) where S is 0, then J is equal to L by the usual rules of addition of angular momenta. In this situation ($S=0$), J parity is redundant to the parity of the nucleon-antinucleon system, since

$$(-1)^J = (-1)^L = -(-1)^{L+1} = -P. \tag{4.4a}$$

The Regge trajectory in question is thus associated with $h_0^{(\pm)T}(J,u)$ only, and $(-1)^T = GP$.

For trajectories of class (b) where S is 1, then J is either $L\pm 1$ or L by the usual rules of addition of angular momenta; this statement also holds for all points on the Regge trajectory coupled to the nucleon-antinucleon channel. In this situation, J parity is not specified by ordinary parity (the parity of the nucleon-antinucleon system), since

$$(-1)^J = (-1)^L = -P \quad \text{for} \quad J = L,$$
and
$$(-1)^J = -(-1)^L = P \quad \text{for} \quad J = L\pm 1. \tag{4.4b}$$

However, Eq. (4.4b) leads to the important result that the value of $(-1)^J P$ determines whether a particular Regge trajectory is associated with $J=L$ triplet or $J=L\pm 1$ triplet partial-wave amplitudes for class (b) trajectories. In particular, for

$$(-1)^J P = -1,$$

the trajectory is associated with $h_1^{(\pm)T}(J,u)$; for

$$(-1)^J P = 1,$$

the trajectory is associated with $h_{11}^{(\pm)T}(J,u)$, $h_{12}^{(\pm)T}(J,u)$, and $h_{22}^{(\pm)T}(J,u)$.

It is worthy of note that in all the foregoing the quantities L and S are specific to the nucleon-antinucleon system, and constraints are imposed on these quantities when the nucleon-antinucleon channel is to have the same quantum numbers as some given Regge trajectory.

Table I. The independent quantum numbers of the $N\bar{N}$ channel.

G	T	P	$(-1)^J$	S	$(-1)^J P$	
(+)	0	(+)	(+)	1	(+)	Vacuum, ABC
(+)	1	(+)	(−)	0	(−)	
(+)	0	(−)	(+)	0	(−)	χ, (η)
(+)	1	(−)	(+)	1	(−)	
(+)	1	(−)	(−)	1	(+)	ρ
(−)	0	(+)	(−)	0	(−)	
(−)	1	(+)	(+)	1	(+)	
(−)	1	(+)	(−)	1	(−)	
(−)	0	(−)	(+)	1	(−)	
(−)	0	(−)	(−)	1	(+)	ω
(−)	1	(−)	(+)	0	(−)	π
(+)	0	(+)	(−)	1	(−)	

To summarize the results of this section, all the "good" quantum numbers of the nucleon-antinucleon channel including J parity, are given in Table I. Also S and $(-1)^J P$ are given so that the amplitudes containing the Regge trajectories can be easily identified. Some particles whose quantum numbers have been indicated experimentally are entered at the far right. From the table it is seen that there are twelve independent sets of quantum numbers. In those cases in which S is 0 (class a), J parity and parity are not independent; this fact reduces the number from 16 to 12. There would be 16 independent sets of J parity and parity were always independent.

5. REGGE POLES IN THE NUCLEON-NUCLEON CHANNEL

This brief section is devoted to the study of Regge trajectories with the quantum numbers of the nucleon-nucleon channel. Such a study is important because the deuteron and the enhancement of the singlet np scattering[18] cross section at threshold can be considered as points on Regge trajectories.

The quantum numbers of the nucleon-nucleon system are

$$\begin{aligned} &\text{(a)}\quad \text{baryon number} = B = 2,\\ &\text{(b)}\quad \text{strangeness} = 0,\\ &\text{(c)}\quad \text{isotopic spin} = T = 0,\,1, \qquad (5.1)\\ &\text{(d)}\quad \text{total spin} = S = 0,\,1,\\ \text{and}\quad &\text{(e)}\quad \text{parity} = P = (-1)^L, \end{aligned}$$

where L is the relative orbital-angular momentum of the nucleon-nucleon system. Not all the above quantum numbers are independent, however. For the scattering of two identical fermions, Pauli symmetry imposes the following constraint for the nucleon-nucleon system:

$$(-1)^{L+S+T} = -1. \tag{5.2}$$

[18] A. O. Barut, Phys. Rev. 126, 1873 (1962).

Once the parity and isotopic spin of the Regge trajectory with the same quantum numbers as the nucleon-nucleon system are specified, the total spin is no longer independent but is determined by

$$(-1)^S = -P(-1)^T. \tag{5.3}$$

And, as in the nucleon-antinucleon channel, there are two classes of trajectories to be considered separately. They are (a) $P(-1)^T=-1$ and (b) $P(-1)^T=+1$. In class (a), $S=0$, and in class (b), $S=1$. The results of Sec. 4 are immediately applicable for the nucleon-nucleon channel. If $P(-1)^T=-1$, $S=0$, and $J=L$, then the trajectory is associated with $h_0^{(\pm)T}(J,s)$. If $P(-1)^T=1$, $S=1$, $J=L$, or $L\pm1$, then there are two possibilities: $P(-1)^J=1$ where the trajectory is associated with $h_1^{(\pm)T}(J,s)$, and $P(-1)^J=-1$ where the trajectory is associated with $h_{11}^{(\pm)T}(J,s)$, $h_{22}^{(\pm)T}(J,s)$, and $h_{12}^{(\pm)T}(J,s)$.

Table II shows the independent quantum numbers of the nucleon-nucleon system together with S and $P(-1)^J$. If one counts properly there are six independent trajectories with the quantum numbers of the nucleon-nucleon channel. This is only half the number for $N\bar{N}$ channel because G parity is not well defined for states with baryon number $=2$.

6. HIGH-ENERGY NUCLEON-NUCLEON SCATTERING

In this section we treat the exchange of systems with the quantum numbers of the nucleon-antinucleon channel as Regge trajectories of classes (a) and (b) mentioned in Sec. 4. In particular, the pion and ρ-meson trajectories are considered, the Pomeranchuk trajectory already having been discussed by a number of authors.[4] The ω and χ trajectories are not considered, since they are isotopic spin 0 and do not have the quantum numbers of the u channel for np scattering that is of primary interest for backward scattering.

A. The Pion Trajectory

The quantum numbers of the pion are such that the trajectory is associated with the partial-wave amplitude

TABLE II. The independent quantum numbers of the NN channel.[a]

T	P	$(-1)^J$	S	$(-1)^JP$	
0	(+)	(+)	1	(+)	
1	(+)	(+)	0	(+)	*
0	(+)	(−)	1	(−)	deuteron
1	(−)	(−)	1	(+)	
1	(−)	(+)	1	(−)	
0	(−)	(−)	0	(+)	

[a] The deuteron has been entered at the appropriate place and the asterisk indicates the enhancement of the singlet np system at threshold. Experimentally, these are the only two trajectories that reach the right-half J plane. As explained by Barut (reference 18) the virtual singlet S state of the NN system corresponds to a trajectory that turns around just before reaching $J=0$.

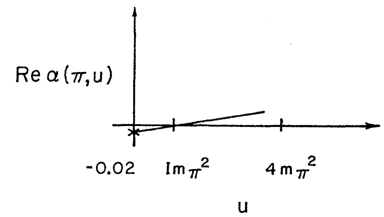

FIG. 3. The π-meson trajectory.

$h_0^{(+)1}(J,u)$ (Sec. 4). From Eq. (3.30) the contribution of the pion trajectory to the invariant scattering amplitude in the u channel is,[17] from Eq. (3.33),

$$\bar{G}_1{}^1(u,s,t) = -\frac{\pi}{2E_u{}^2}\frac{\tilde{\beta}_{0,\pi}{}^{(+)1}(u)[2\alpha(\pi)+1]}{\sin\pi\alpha(\pi)}\left(\frac{s}{s_0}\right)^{\alpha(\pi)} \times[1+\exp i\pi\alpha(\pi)], \tag{6.1}$$

$$\bar{G}_1{}^0=\bar{G}_2{}^1=\bar{G}_2{}^0=\bar{G}_3{}^0=\bar{G}_4{}^1=\bar{G}_4{}^0=\bar{G}_5{}^1=\bar{G}_5{}^0=0,$$

where $\alpha(\pi)$ is the position of the Regge pole for the pion. Following the work of Chew and Frautschi[3] we assume that $\mathrm{Re}\,\alpha$ has a positive slope $\cong 1/50m_\pi{}^2$ as shown in Fig. 3. Note that $z_u=-1-(s/2p_u{}^2)$ in Eq. (3.33).

The amplitude $\bar{G}_1{}^1(u,s,t)$ satisfies the Mandelstam representation and has only the singularities required by unitarity; hence, $\tilde{\beta}_{0,\pi}{}^{(+)1}(u)$ has a zero of order $E_u{}^2$ at $E_u{}^2=0$ so that Eq. (6.1) be finite at $E_u{}^2=0$.

Using the crossing matrix (2.24)′, the contribution of the single-pion exchange to nucleon-nucleon scattering is

$$\begin{aligned}
G_1{}^{(0,1)}(s,u,t)&=-\tfrac{1}{4}\bar{G}_1{}^1(u,s,t)\tfrac{1}{2}(3,1),\\
G_2{}^{(0,1)}(s,u,t)&=\tfrac{1}{4}\bar{G}_1{}^1(u,s,t)\tfrac{1}{2}(3,1),\\
G_3{}^{(0,1)}(s,u,t)&=\tfrac{1}{4}\bar{G}_1{}^1(u,s,t)\tfrac{1}{2}(3,1),\\
G_4{}^{(0,1)}(s,u,t)&=-\tfrac{1}{4}\bar{G}_1{}^1(u,s,t)\tfrac{1}{2}(3,1),
\end{aligned} \tag{6.2}$$

and

$$G_5{}^{(0,1)}(s,u,t)=-\tfrac{1}{4}\bar{G}_1{}^1(u,s,t)\tfrac{1}{2}(3,1).$$

From Eq. (2.20) the contribution to the "physical amplitudes" of nucleon-nucleon scattering is

$$\begin{aligned}
E\phi_2{}^{(0,1)}(W,z)&=-\frac{\pi}{4}\frac{\tilde{\beta}_{0,\pi}{}^{(+)1}(u)[2\alpha(\pi)+1]}{\sin\pi\alpha(\pi)}\left(\frac{s}{s_0}\right)^{\alpha(\pi)}\times[1+\exp i\pi\alpha(\pi)]\tfrac{1}{2}(3,1),\\
E\phi_3{}^{(0,1)}(W,z)&=-\frac{\pi}{4}\frac{\tilde{\beta}_{0,\pi}{}^{(+)1}(u)[2\alpha(\pi)+1]}{\sin\pi\alpha(\pi)}\left(\frac{s}{s_0}\right)^{\alpha(\pi)}\times[1+\exp i\pi\alpha(\pi)]\tfrac{1}{2}(3,1),
\end{aligned} \tag{6.3}$$

and $\phi_1=\phi_4=\phi_5=0$.

Note that this amplitude will vanish as u approaches zero because of the residue $\tilde{\beta}_0{}^{(+)1}(u)$. The vanishing of the amplitude at $u=0$ (backward direction for nucleon-nucleon scattering) will be true for all the class (a) trajectories of the nucleon-antinucleon u channel.

Similarly, using Pauli symmetry (2.21) and the crossing matrix, (2.24)′, we can calculate the contribution of the single-pion exchange in the t channel to nucleon-nucleon scattering; the result is

$$E\phi_2^{(0,1)}(W,z)\sim-\frac{\pi}{4}\frac{\tilde{\beta}_{0,\pi}{}^{(+)1}(t)[2\alpha(\pi)+1]}{\sin\pi\alpha(\pi)}\left(\frac{s}{s_0}\right)^{\alpha(\pi)}\times[1+\exp i\pi\alpha(\pi)]\tfrac{1}{2}(3,-1)$$

and

$$E\phi_4^{(0,1)}(W,z)\sim-\frac{\pi}{4}\frac{\tilde{\beta}_{0,\pi}{}^{(+)1}(t)[2\alpha(\pi)+1]}{\sin\pi\alpha(\pi)}\left(\frac{s}{s_0}\right)^{\alpha(\pi)}\times[1+\exp i\pi\alpha(\pi)]\tfrac{1}{2}(3,-1),\tag{6.4}$$

where $\phi_1=\phi_3=\phi_5=0$. Note that the single pion of the nucleon-antinucleon t channel will give zero contribution to the nucleon-nucleon amplitude in the forward direction.

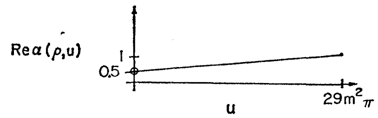

FIG. 4. The ρ-meson trajectory.

B. The ρ-Meson Trajectory

The quantum numbers of the ρ meson are such that the trajectory is associated with the amplitudes $h_{11}{}^{(-)1}(J,u)$, $h_{22}{}^{(-)1}(J,u)$, and $h_{12}{}^{(-)1}(J,u)$ (Sec. 4). The contribution of the ρ meson to the invariant amplitudes $\bar{G}$ asymptotically in z_u is:

$$\bar{G}_1{}^1(u,s,t)\sim-\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\frac{1}{E_u{}^2}\left[-\beta_{22,\rho}{}'^{(-)1}(u)\alpha(\rho)+\frac{2E_u{}^2}{m^2}\beta_{12,\rho}{}'^{(-)1}(u)\right]z_u{}^{\alpha(\rho)}\{1-\exp[-i\pi\alpha(\rho)]\},$$

$$\bar{G}_2{}^1(u,s,t)\sim-\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\frac{1}{p_u{}^2}\left[-\beta_{22,\rho}{}'^{(-)1}(u)\alpha(\rho)+\frac{2E_u{}^2}{m^2}\beta_{12,\rho}{}'^{(-)1}(u)\right]z_u{}^{\alpha(\rho)-1}\{1-\exp[-i\pi\alpha(\rho)]\},$$

$$\bar{G}_3{}^1(u,s,t)\sim0,\tag{6.5}$$

$$\bar{G}_4{}^1(u,s,t)\sim\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\frac{1}{p_u{}^2}[-\beta_{22,\rho}{}'^{(-)1}(u)\alpha(\rho)+2\beta_{12,\rho}{}'^{(-)1}(u)]z_u{}^{\alpha(\rho)-1}\{1-\exp[-i\pi\alpha(\rho)]\},$$

$$\bar{G}_5{}^1(u,s,t)\sim-\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\frac{1}{p_u{}^2}\left\{-\beta_{11,\rho}{}'^{(-)1}(u)[2\alpha(\rho)+1]-\beta_{22,\rho}{}'^{(-)i}(u)\alpha(\rho)+\frac{2(E_u{}^2+m^2)}{m^2}\beta_{12,\rho}{}'^{(-)1}(u)\right\}\times z_u{}^{\alpha(\rho)}\{1-\exp[-i\pi\alpha(\rho)]\}.$$

The quantity $\alpha(\rho)$ determines the trajectory of the ρ meson; the real part of $\alpha(\rho)$, $\mathrm{Re}\alpha(\rho)$, is equal to 1 at $u=29m_\pi{}^2$, and is illustrated in Fig. 4. From Eqs. (2.24)′ and (2.20), the contribution of the ρ meson to the "physical amplitudes" of nucleon-nucleon scattering is

$$2E\phi_1{}^{(0,1)}(W,z)=\left[m^2(\bar{G}_2{}^1-\bar{G}_5{}^1)-(p^2+E^2)\bar{G}_4{}^1-\frac{um^2}{4p^2}(\bar{G}_2{}^1+\bar{G}_4{}^1+\bar{G}_5{}^1)\right]\tfrac{1}{2}(3,1),$$

$$2E\phi_2{}^{(0,1)}(W,z)=\left\{-2(p^2+E^2)\bar{G}_2{}^1-\frac{u}{4p^2}[m^2\bar{G}_4{}^1+E^2\bar{G}_5{}^1+(p^2+E^2)\bar{G}_2{}^1]-\frac{u}{4}\bar{G}_1{}^1\right\}\tfrac{1}{2}(3,1),$$

$$2E\phi_3{}^{(0,1)}(W,z)=\left[-\frac{u}{2p^2}\left(\frac{m^2}{2}(\bar{G}_2{}^1+\bar{G}_4{}^1)+\frac{E^2}{2}\bar{G}_5{}^1\right)+\frac{u}{4}\bar{G}_1{}^1\right]\tfrac{1}{2}(3,1),\tag{6.6}$$

$$2E\phi_4{}^{(0,1)}(W,z)=-\frac{t}{2p^2}\left[\frac{(p^2+E^2)\bar{G}_4{}^1}{2}+\frac{m^2}{2}(\bar{G}_2{}^1+\bar{G}_5{}^1)\right]\tfrac{1}{2}(3,1),$$

and

$$2m\phi_5{}^{(0,1)}(W,z)=-\frac{m^2(ut)^{1/2}}{4p^2}(\bar{G}_2{}^1+\bar{G}_4{}^1+\bar{G}_5{}^1)\tfrac{1}{2}(3,1).$$

Keeping only the leading terms in $s=-2p_u^2(1+z_u)$ and $(u/2p^2)$ $(2p^2=mT_L\gtrsim 100m_\pi^2$, where T_L is the lab system kinetic energy, $T_L\gtrsim 3$ BeV), we have

$$2E\phi_1^{(0,1)}(W,z)=-2E\phi_4^{(0,1)}(W,z)\sim -\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\times\left\{-[2\alpha(\rho)+1]\tilde{\beta}_{11,\rho}^{(-)1}(u)+\frac{u}{m^2}\tilde{\beta}_{12,\rho}^{(-)1}(u)-\frac{u}{4m^2}\tilde{\beta}_{22,\rho}^{(-)1}(u)\alpha(\rho)\right\}\times\left(\frac{s}{s_0}\right)^{\alpha(\rho)}[1-\exp i\pi\alpha(\rho)]\tfrac{1}{2}(3,1),$$

$$2E\phi_2^{(0,1)}(W,z)=2E\phi_3^{(0,1)}(W,z)\sim\frac{\pi}{2}\frac{1}{\sin\pi\alpha(\rho)}\left[-\tilde{\beta}_{22,\rho}^{(-)1}(u)\alpha(\rho)+\frac{u}{m^2}\tilde{\beta}_{12,\rho}^{(-)1}(u)-\frac{u}{4m^2}[2\alpha(\rho)+1]\tilde{\beta}_{11,\rho}^{(-)1}(u)\right]\times\left(\frac{s}{s_0}\right)^{\alpha(\rho)}[1-\exp i\pi\alpha(\rho)]\tfrac{1}{2}(3,1),\quad(6.7)$$

and

$$2m\phi_5^{(0,1)}(W,z)\sim -\frac{\pi}{2}\frac{(ut)^{1/2}}{4p^2}\frac{1}{\sin\pi\alpha(\rho)}\left\{-[2\alpha(\rho)+1]\tilde{\beta}_{11,\rho}^{(-)1}(u)-\tilde{\beta}_{22,\rho}^{(-)1}(u)+2\tilde{\beta}_{12,\rho}^{(-)1}(u)+\frac{u}{2m^2}\tilde{\beta}_{12,\rho}^{(-)1}(u)\right\}\times\left(\frac{s}{s_0}\right)^{\alpha(\rho)}[1-\exp i\pi\alpha(\rho)]\tfrac{1}{2}(3,1).$$

Similarly, using Pauli symmetry (2.21) and the crossing matrix (2.24)′, we find the contribution of the Regge trajectories in the t channel with the quantum numbers of class (b) (Sec. to 4) the nucleon-nucleon channel to be

$$2E\phi_1^T(W,z)=2E\phi_3^T(W,z)\sim -\frac{\pi}{2}\sum_{T'}\frac{(-1)^T B^{TT'}}{\sin\pi\alpha}\times\left[-(2\alpha+1)\tilde{\beta}_{11}^{(\pm)T'}(t)+\frac{t}{m^2}\tilde{\beta}_{12}^{(\pm)T'}(t)-\frac{t}{4m^2}\tilde{\beta}_{22}^{(\pm)T'}(t)\alpha\right]\left(\frac{s}{s_0}\right)^{\alpha}(\exp i\pi\alpha\pm 1),$$

$$2E\phi_2^T(W,z)=-2E\phi_4^T(W,z)\sim\frac{\pi}{2}\sum_{T'}\frac{(-1)^T B^{TT'}}{\sin\pi\alpha}\times\left[-\tilde{\beta}_{22}^{(\pm)T'}(t)\alpha+\frac{t}{m^2}\tilde{\beta}_{12}^{(\pm)T'}(t)-\frac{t}{4m^2}(2\alpha+1)\tilde{\beta}_{11}^{(\pm)T'}(t)\right]\left(\frac{s}{s_0}\right)^{\alpha}(\exp i\pi\alpha\pm 1),\quad(6.8)$$

and

$$2E\phi_5^T(W,z)\sim\frac{\pi}{2}\frac{(ut)^{1/2}}{4p^2}\sum_{T'}\frac{(-1)^T B^{TT'}}{\sin\pi\alpha}\times\left[-(2\alpha+1)\tilde{\beta}_{11}^{(\pm)T'}-\tilde{\beta}_{22}^{(\pm)T'}\alpha+2\tilde{\beta}_{12}^{(\pm)T'}+\frac{t}{2m^2}\tilde{\beta}_{12}^{(\pm)T'}\right]\left(\frac{s}{s_0}\right)^{\alpha}(\exp i\pi\alpha\pm 1),$$

where $(-1)^T B^{TT'}$ is the isotopic spin crossing matrix

$$\tfrac{1}{2}\begin{bmatrix}-1 & 3\\ -1 & -1\end{bmatrix}.$$

The total cross section is given by the optical theorem:

$$\sigma=(4\pi/p)\,\mathrm{Im}[\mathrm{Tr}U\phi(W,0)]\quad(6.9)$$

in matrix notation, where U is the density matrix for the incident beam. For unpolarized incident beam,

$$U=\delta_{\lambda\lambda'}\delta_{\mu\mu'}/4,\quad(6.10)$$

and

$$\sigma=(\pi/p)\,\mathrm{Im}\sum_{\mu\mu'}\sum_{\lambda\lambda'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\phi_{\lambda'\mu',\lambda\mu}(W,0)=(2\pi/p)\,\mathrm{Im}[\phi_1(W,0)+\phi_3(W,0)].\quad(6.11)$$

For example, from (6.8), the contribution of the ρ

meson to the np and pp total cross sections is

$$\sigma_{pp}=\frac{\pi^2}{2pE}[2\alpha(\rho,0)+1]\tilde{\beta}_{11,\rho}{}^{(-)1}(0)\left(\frac{s}{s_0}\right)^{\alpha(\rho,0)},$$

and (6.12)

$$\sigma_{np}=-\frac{\pi^2}{2pE}[2\alpha(\rho,0)+1]\tilde{\beta}_{11,\rho}{}^{(-)1}(0)\left(\frac{s}{s_0}\right)^{\alpha(\rho,0)}.$$

The difference between the above cross sections contains the quantum numbers of the ρ and π mesons in the t channel; however, the π meson does not contribute to the forward amplitude in Eq. (6.11), and

$$\sigma_{pp}-\sigma_{np}=\frac{\pi^2}{pE}[2\alpha(\rho,0)+1]\tilde{\beta}_{11,\rho}{}^{(-)1}(0)\left(\frac{s}{s_0}\right)^{\alpha(\rho,0)}. \quad (6.13)$$

The differential cross section per unit center-of-mass solid angle is given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{4}\sum_{\lambda'\lambda\mu'\mu}|\phi_{\lambda'\mu'\lambda\mu}|^2$$
$$=\tfrac{1}{2}(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2+|\phi_4|^2+4|\phi_5|^2) \quad (6.14)$$

for an unpolarized incident beam. The differential cross section per unit momentum transfer, $\Delta^2=-u$, is related to the differential cross section per unit center-of-mass solid angle by

$$d\sigma/d\Delta^2=(\pi/p^2)d\sigma/d\Omega. \quad (6.15)$$

Thus, the contribution of the ρ meson to the np differential cross section near the backward direction is from Eqs. (6.7) and (6.14) (keeping only the leading terms in s and the linear terms in Δ^2)

$$\frac{d\sigma}{d\Delta^2}=\frac{\pi^3}{16p^2E^2}\left|\left(\frac{s}{s_0}\right)^{\alpha(\rho)}\right|^2\left\{|[2\alpha(\rho)+1]\tilde{\beta}_{11,\rho}{}^{(-)1}(-\Delta^2)|^2+|\tilde{\beta}_{22,\rho}{}^{(-)1}(-\Delta^2)\alpha(\rho)|^2+\frac{2\Delta^2}{m^2}|\tilde{\beta}_{12,\rho}{}^{(-)1}(-\Delta^2)|^2\right\}\times\frac{1}{\{\cos[\pi\alpha(\rho)/2]\}^2}, \quad (6.16)$$

where $\alpha(\rho)=\alpha(\rho,u)=\alpha(\rho,-\Delta^2)$. For $\Delta^2\ll 2m^2$, we have

$$\frac{d\sigma}{d\Delta^2}\approx\frac{\pi^3}{16p^2E^2}\left|\left(\frac{s}{s_0}\right)^{\alpha(\rho)}\right|^2\times\frac{|\tilde{\beta}_{11,\rho}{}^{(-)1}(-\Delta^2)[2\alpha(\rho)+1]|^2}{\{\cos[\pi\alpha(\rho)/2]\}^2}. \quad (6.17)$$

The combination of residues $\beta_{22}\alpha+(\Delta^2/2m^2)\beta_{12}$ drops out for small Δ^2 since this combination has a zero of order Δ^2 at $\Delta^2=0$ from Eq. (6.5). Equation (6.17) will be useful for the analysis of the energy dependence of the backward peak in np scattering; from a study of the energy dependence of $d\sigma/d\Delta^2$ for small fixed Δ^2 ($\Delta^2\ll 2m^2$), $\alpha(\rho,\Delta^2)$ can be determined in the region of Δ^2 equal to zero. Equation (6.16) will be useful for a study of the dependence of the combination of residues in the curly bracket upon Δ^2 as Δ^2 is increased. The pion trajectory has not been included in Eq. (6.16), since the pion trajectory lies below the ρ trajectory and dominates the high-energy differential cross section to a lesser degree.

The quantity $\beta(u)/(p_u{}^2)^\alpha$ is real for $u<0$[19]; the ratio of the real parts of the amplitudes in (6.7) to the imaginary parts has the definite value of $\tan(\pi\alpha/2)$. The imaginary part of ϕ_1 and ϕ_4 for np scattering for $u=\Delta^2=0$ is

$$\frac{\pi}{4E}\tilde{\beta}_{11}{}^{(-)1}(0)[2\alpha(\rho,0)+1]\left(\frac{s}{s_0}\right)^{\alpha(\rho,0)}, \quad (6.18)$$

which is given by the optical theorem in the difference of the pp and np total cross sections, Eq. (6.13). Also by use of Eqs. (6.17) and (6.13), this difference of total cross sections is related to the np differential cross section by

$$(\sigma_{pp}-\sigma_{np})^2=16\pi\cos^2\left[\frac{\pi\alpha(\rho,0)}{2}\right]\left(\frac{d\sigma_{np}}{d\Delta^2}\right)\Bigg|_{\Delta^2=0}. \quad (6.19)$$

7. CONCLUSIONS

The high-energy nucleon-nucleon problem has been considered from the point of view of the Regge trajectories with the quantum numbers of the nucleon-antinucleon channel. In particular, the analyticity of the partial-wave amplitude in total angular momentum was discussed, and a unique continuation was found. In the last section, the contribution of the ρ- and π-meson trajectories to NN scattering was found, and the resulting formulas (6.16), (6.17), and (6.19) should be useful for an analysis of the np differential cross section near the backward direction for energies greater than $\approx$3 BeV.[20]

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[19] Philip Burke (private communication).

[20] Upon completion of this work the author was informed that the problem of Regge poles in the $N-N$ and $N-\bar{N}$ amplitudes has been studied by M. Gell-Mann and David Sharp of the California Institute of Technology, and V. N. Gribov and D. V. Volkov of the Ioffe Physico-Technical Institute, Leningrad, U.S.S.R. In the latter work the possibility of coincidence of Regge poles at $t=0$ (t being the energy of the nucleon-antinucleon t channel) was considered.